

A strong converse for the quantum state merging protocol

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Abstract

The Polyanskiy-Verdú paradigm provides an elegant way of using generalized-divergences to obtain strong converses and thus far has remained confined to protocols involving channels (classical or quantum). In this paper, drawing inspirations from it, we provide strong converses for protocols involving LOCC (local operations and classical communication). The key quantity that we work with is the Rényi relative entropy of entanglement. We provide a strong converse for the quantum state merging protocol that gives an exponential decay of the fidelity of the protocol for rates below the optimum with the number of copies of the state and are provided both for entanglement rate with LOCC as well as for classical communication with one-way LOCC. As an aside, the developments also yield short strong converses for the entanglement-concentration of pure states and the Schumacher compression.

1 Introduction

Many information-theoretic problems deal with finding out the minimal resources needed to accomplish a task or the maximal yield obtained after a task given the constraints and many times a definite optimal answer is provided when the number of copies of the input(s), resources, or output(s) is large and examples include Schumacher compression, quantum state merging, entanglement concentration etc. [1, 2, 3, 4].

These answers are typically given in two parts: for the number of resources larger (respectively yield smaller) than the optimal, there exists a protocol to accomplish the task (termed as achievability) and for the number of resources smaller (respectively yield larger) than the optimal, any protocol will perform badly (termed as converse). In the latter case, a strong converse, if it exists, additionally says that any protocol will perform very badly (as bad as it can be) and this is quantified in terms of a performance measure. Strong converses provide

a refined view to the optimal quantities since now they can be seen as a sharp dividing line between what can and cannot be achieved.

These strong converses have a long history starting from the works in the classical case of Wolfowitz [5], Arimoto [6] to more recent works in the quantum case of Winter [7, 8], Ogawa & Nagaoka [9] and König and Wehner [10]. This is hardly a list of exhaustive references on the topic since there have been more recent works (some of which are referenced later) and the literature on smooth entropies starting from the work of Renner and Wolf [11] provides bounds that when coupled with asymptotic equipartition property for the independent and identically distributed (i.i.d.) copies yield strong converses as well.

Recently, a fresh and enchanting take on strong converses has been provided by Polyanskiy and Verdú using the generalized relative entropies that satisfy certain well expected properties such as monotonicity under the application of the classical channels [12].

The idea of using monotonicity to prove converses is due to Blahut who used it to prove the Fano inequality that gives the weak converse [13]. Blahut employed the traditional relative entropy as opposed to the generalized relative entropies employed by Polyanskiy and Verdú who show that, in particular, the Rényi relative entropy would yield the Arimoto converse.

Arimoto's proof was extended by Ogawa & Nagaoka for sending classical information across quantum channels. But it has not been possible to extend Arimoto's proof for other protocols such as getting an exponential bound for the quantum information transfer across quantum channels. Unlike Arimoto's proof, Polyanskiy and Verdú's proof relied on certain properties of generalized relative entropies that are also satisfied by the quantum generalized relative entropies and their approach has now been extended to the quantum domain (see Ref. [14]) and applied to various protocols [15, 16] (see also Ref. [15] for a discussion on this).

Not all protocols admit strong converses [17] and in some cases, no definite answer is known whether a strong converse would exist or not. For example, we don't have a strong converse for the quantum capacity except in some special cases mentioned in Ref. [18] although the same paper provides a "pretty strong converse" for degradable channels.

The Polyanskiy-Verdú paradigm has thus far remained confined to protocols involving channels. Quantum information theory is richly endowed with another class of operations namely the LOCC. Inspired by the paradigm, could one address the protocols involving LOCC?

Just as Polyanskiy and Verdú define Rényi mutual information that does not increase under the application of the channel (see the quantum Rényi information measures in Ref. [14]), we seek a Rényi entanglement measure that does not increase under LOCC. The relative entropy of entanglement is an entanglement monotone [19, 20] and hence, does not increase under LOCC. A Rényi relative entropy of entanglement is then not hard to define that, leveraging the result by Vedral *et al* [21], does not increase under LOCC.

For the protocols involving channels, typically, there is a reduction, using monotonicity, to Rényi relative entropy involving binary distributions that are a functions of probability of error or the fidelity that tell us how well the protocol performs. Such reductions no longer

seem to be applicable/useful for protocols such as the quantum state merging. Instead an inequality by van Dam and Hayden [22] involving the Rényi entropies of two states and their fidelity given in a completely different context turns out to be just the thing one is looking for and it is also useful in providing bounds for the Rényi relative entropy of entanglement.

For a Hilbert space \mathcal{H} , we define $\mathcal{S}(\mathcal{H}) = \{\rho \geq 0 : \text{Tr}\rho = 1\}$. Fidelity F between the states $\rho, \sigma \in \mathcal{S}(\mathcal{H})$ is $F(\rho, \sigma) \equiv \|\sqrt{\rho}\sqrt{\sigma}\|_1 = \max |\langle \phi | \psi \rangle|$, where the maximization is over all purifications $|\phi\rangle, |\psi\rangle$ of ρ and σ respectively.

We denote a maximally entangled state in $\mathcal{H}_A \otimes \mathcal{H}_B$ with Schmidt rank K by Φ_K^{AB} . The set of bipartite separable states of A and B is denoted by $\mathcal{S}^{A:B}$. The terms quantum operation and completely positive trace preserving (cptp) map are used interchangeably. For a pure state $|\Psi\rangle$, $\Psi = |\Psi\rangle\langle\Psi|$.

All the logarithms are to the base 2. The quantum relative entropy from ρ to σ is given by $S(\rho||\sigma) \equiv \text{Tr}\rho(\log\rho - \log\sigma)$, the von Neumann entropy is given by $S(A)_\rho \equiv -\text{Tr}\rho^A \log\rho^A$. At times, we shall use $S_\alpha(\rho^A)$ instead of $S_\alpha(A)_\rho$. For a bipartite state ρ^{AB} , the conditional entropy of A given B is given by $S(A|B)_\rho \equiv S(AB)_\rho - S(B)_\rho$ and the quantum mutual information between A and B is given by $I(A : B)_\rho \equiv S(A)_\rho + S(B)_\rho - S(AB)_\rho$.

For $\rho, \sigma \geq 0$, $\alpha \in [0, 2] \setminus \{1\}$, the α -quasi-relative entropy, from ρ to σ ($\rho, \sigma \geq 0$) is defined as

$$Q_\alpha(\rho||\sigma) \equiv \text{sign}(\alpha - 1)\text{Tr}\rho^\alpha\sigma^{1-\alpha}, \quad (1)$$

and the Rényi α -relative entropy from ρ to σ is defined as

$$S_\alpha(\rho||\sigma) \equiv \frac{1}{\alpha - 1} \log \text{Tr}\rho^\alpha\sigma^{1-\alpha} \quad (2)$$

where limits are taken for $\alpha = 1$ and we drop the subscript. One can derive this from the generalized divergences defined by Petz [23]. We quickly recall from Refs. [23, 24] that for $\alpha \in [0, 2]$ and a cptp map \mathcal{E} ,

$$S_\alpha(\rho||\sigma) \geq S_\alpha[\mathcal{E}(\rho)||\mathcal{E}(\sigma)]. \quad (3)$$

Let $S_\alpha(A)_\rho$, $\alpha \geq 0$ be the α -entropy of ρ^A given by

$$S_\alpha(A)_\rho = \frac{1}{1 - \alpha} \log \text{Tr}(\rho^A)^\alpha, \quad (4)$$

where, again, limits are taken for $\alpha = 1$ and we drop the subscript. The Rényi coherent information (see Ref. [14]) is given by

$$I_\alpha(A|B)_\rho \equiv \frac{\alpha}{\alpha - 1} \log \text{Tr} [\text{Tr}_B(\rho^{AB})^\alpha]^{1/\alpha}. \quad (5)$$

2 Nature of the bounds obtained

The strong converse bounds we provide are obtained using the Rényi relative entropy similar in spirit to that of Arimoto and Polyanskiy & Verdú [6, 12]. These bound are provided for

the quantum state merging, entanglement concentration and the Schumacher compression. We note that strong converse is already known for these protocols but, to the best of author's knowledge, there are no strong converses known using the Rényi approach.

We now illustrate the nature of the bounds we provide. Suppose for a bipartite state ρ^{AB} , the optimal quantity is given in terms of a function $f(\rho^{AB})$. Let $\mathcal{I} \subseteq \mathbb{R}$ be an interval with 1 as its boundary point and not containing $\{0\}$. Suppose a Rényi generalization of $f(\rho^{AB})$ for $\alpha \in \mathcal{I}$ is given by $f_\alpha(\rho^{AB})$ with $f_\alpha(\rho^{AB}) \leq f(\rho^{AB})$ for $\alpha \in \mathcal{I}$, where $\lim_{\alpha \rightarrow 1} f_\alpha(\rho^{AB}) = f(\rho^{AB})$ and for n copies, $f_\alpha[(\rho^{AB})^{\otimes n}] = n f_\alpha(\rho^{AB})$. Suppose the resources consumed that we want to lower bound are $g(n)$ for n copies. The bounds that we obtain are of the form

$$\log(\text{Fidelity of the protocol}) \leq n\zeta \left| \frac{\alpha - 1}{\alpha} \right| \left[\frac{g(n)}{n} - f_\alpha(\rho^{AB}) \right], \quad (6)$$

where $\zeta > 0$ is a constant not dependent on any parameters. These bounds clearly have the same flavor as the Arimoto converse [6]. If for all n , $g(n)/n$ is bounded from above by R such that $R < f(\rho^{AB})$, then it follows that we can choose a α close to 1 such that $g(n)/n - f_\alpha(\rho^{AB}) \leq R - f_\alpha(\rho^{AB})$ is negative and the RHS is independent of n .

Note that in some cases, instead of resources consumed we are interested in the yield $h(n)$ (for n copies). For example, in the case of entanglement concentration, we are interested in the number of EPR pairs generated. Then the bounds obtained are of the form

$$\log(\text{Fidelity of the protocol}) \leq n\zeta \left| \frac{\alpha - 1}{\alpha} \right| \left[f_\alpha(\rho^{AB}) - \frac{h(n)}{n} \right]. \quad (7)$$

In this case, $f_\alpha(\rho^{AB}) \geq f(\rho^{AB})$ with $\lim_{\alpha \rightarrow 1} f_\alpha(\rho^{AB}) = f(\rho^{AB})$. If $h(n)/n \geq R$ and $R > f(\rho^{AB})$, then $f_\alpha(\rho^{AB}) - h(n)/n \leq f_\alpha(\rho^{AB}) - R$ and hence, we can choose an α such that $f_\alpha(\rho^{AB}) - R$ is negative and the RHS is independent of n .

In either case, the fidelity decays exponentially with n . Our purpose of stating this 'last mile' common to several protocols is to avoid repetition and we shall henceforth state a strong converse as the bounds in (6) or (7).

3 Rényi relative entropy of entanglement

We define the Rényi α -relative entropy of entanglement (RREE) for a bipartite state ρ^{AB} as

$$E_{RE}^{(\alpha)}(A : B)_\rho \equiv \inf_{\sigma^{AB} \in \mathcal{S}_{A:B}} S_\alpha(\rho^{AB} || \sigma^{AB}). \quad (8)$$

We now prove some of its properties.

Lemma 1. *For any cptp map $\mathcal{E} : B \rightarrow C$ and $\alpha \in [0, 2]$,*

$$E_{RE}^{(\alpha)}(A : B)_\rho \geq E_{RE}^{(\alpha)}(A : C)_{\mathcal{E}(\rho)}. \quad (9)$$

(Note that the same applies to a local map over A for $E_{RE}^{(\alpha)}$ as well due to symmetry.)

Proof follows from monotonicity using arguments similar to those in Refs. [12, 14] and we omit the details.

Lemma 2 (Vedral *et al* [21]). *RREE is LOCC-monotone, i.e., it does not increase under LOCC.*

We now show that for $\alpha \in (1, 2]$, RREE satisfies a stronger condition than LOCC monotonicity (see Sec XV.B.1 in Ref. [20] and also Refs. [19, 25])). More specifically, this condition states that RREE does not increase on average under the action of LOCC.

Lemma 3. *For $\alpha \in (1, 2]$, for any quantum state ρ^{AB} and any unilocal quantum instrument performed without loss of generality over subsystem A ($\mathcal{E}_k : A \rightarrow A'$) - the \mathcal{E}_k are completely positive maps and their sum is trace preserving - and orthogonal states $\{|k\rangle\langle k|^X\}$, and $\tau^{XA'B} = \sum_x p_k |k\rangle\langle k|^X \otimes \theta_k^{A'B}$, $p_k = \text{Tr} \mathcal{E}_k(\rho^{AB})$, $\theta_k^{A'B} = \mathcal{E}_k(\rho^{AB})/p_k$, we have*

$$E_{RE}^{(\alpha)}(A : B)_\rho \geq E_{RE}^{(\alpha)}(XA' : B)_\tau \geq \sum_k p_k E_{RE}^{(\alpha)}(A' : B)_{\theta_k}. \quad (10)$$

Proof. The first inequality follows from Lemma 1 where the local cptp map ($A \rightarrow XA'$) in question is $\sum_k |k\rangle\langle k|^X \otimes \mathcal{E}_k$. To make the notation clear, let $\mathcal{E}_k(\rho^{AB}) = \sum_j E_{jk} \rho^{AB} E_{jk}^\dagger$, where $\sum_{j,k} E_{jk}^\dagger E_{jk} = \mathbb{1}$. Then the Kraus operators of the above cptp map are $\{|k\rangle\langle k|^X \otimes E_{jk}\}$.

We now prove the second inequality. Define a quantum operation $\mathcal{D} : XA'B \rightarrow XA'B$ with the Kraus operators $\{|k\rangle\langle k|^X \otimes \mathbb{1}^{A'B}\}$. We note that the output of \mathcal{D} for any input is a cq (classical quantum) state which is classical on X and \mathcal{D} does not alter the state $\tau^{XA'B}$. Furthermore, for any $\sigma^{XA'B} \in \mathcal{S}^{XA':B}$, if $\mathcal{D}(\sigma^{XA'B}) = \sum_k q_k |k\rangle\langle k|^X \otimes \sigma_k^{A'B}$, where $\{q_k\}$ is a probability vector, then $\sigma_k^{A'B} \in \mathcal{S}^{A':B}$ for all k . We now have

$$\begin{aligned} E_{RE}^{(\alpha)}(XA' : B)_\tau &= \inf_{\sigma^{XA'B} \in \mathcal{S}^{XA':B}} S_\alpha(\tau^{XA'B} || \sigma^{XA'B}) \end{aligned} \quad (11)$$

$$\geq \inf_{\sigma^{XA'B} \in \mathcal{S}^{XA':B}} S_\alpha \left[\mathcal{D}(\tau^{XA'B}) || \mathcal{D}(\sigma^{XA'B}) \right] \quad (12)$$

$$\geq \inf_{\{q_k\}, \{\sigma_k^{A'B} \in \mathcal{S}^{A':B}\}} S_\alpha \left[\tau^{XA'B} || \sum_k q_k |k\rangle\langle k|^X \otimes \sigma_k^{A'B} \right] \quad (13)$$

$$= \inf_{\{q_k\}, \{\sigma_k^{A'B} \in \mathcal{S}^{A':B}\}} \frac{1}{\alpha - 1} \log \left[\sum_k p_k \left(\frac{p_k}{q_k} \right)^{\alpha-1} Q_\alpha(\theta_k^{A'B} || \sigma_k^{A'B}) \right] \quad (14)$$

$$\geq \inf_{\{q_k\}, \{\sigma_k^{A'B} \in \mathcal{S}^{A':B}\}} \left[S(\underline{p} || \underline{q}) + \sum_k p_k S_\alpha(\theta_k^{A'B} || \sigma_k^{A'B}) \right] \quad (15)$$

$$\geq \sum_k p_k E_{RE}^{(\alpha)}(A' : B)_{\theta_k}, \quad (16)$$

where the first inequality follows because of monotonicity under cptp maps, the second inequality follows because we are minimizing over a bigger set, the third inequality follows

because of the concavity of the logarithm, $S(\underline{p}||\underline{q}) = \sum_k p_k \log(p_k/q_k)$ is the classical relative entropy from probability vectors \underline{p} to \underline{q} , and is non-negative and zero if and only if $\underline{p} = \underline{q}$ (see Ref. [1]), and the last inequality follows by choosing the minimizing \underline{q} and minimizing $\{\sigma_k^{AB}\}$. \square

We now generalize Theorem 4.7 in Ref. [26].

Lemma 4. *For any bipartite state ρ^{AB} and $\alpha \in [0, 2] \setminus \{1\}$, we have*

$$E_{RE}^{(\alpha)}(A : B)_\rho \geq \max \{I_\alpha(A)B)_\rho, I_\alpha(B)A)_\rho\}. \quad (17)$$

For a pure state $|\Psi\rangle^{AB}$, we have

$$S_{1/\alpha}(A)_\Psi \leq E_{RE}^{(\alpha)}(A : B)_\Psi \leq S_{2-\alpha}(A)_\Psi. \quad (18)$$

Proof. We first prove (17). We first note that for a separable state $\sigma^{AB} \in \mathcal{S}^{A:B}$, $\sigma^{AB} \leq \sigma^A \otimes \mathbb{1}^B$.

For $\alpha \in (1, 2]$, invoke the operator monotonicity of $x \mapsto x^{\alpha-1}$, and the fact that $A \geq B$ implies $A^{-1} \leq B^{-1}$ (replacing inverses by generalized inverses if the matrices are singular) to have $(\sigma^{AB})^{1-\alpha} \geq (\sigma^A)^{1-\alpha} \otimes \mathbb{1}^B$. For $\alpha \in [0, 1)$, invoke the operator monotonicity of $x \mapsto x^\alpha$ to have $(\sigma^{AB})^{1-\alpha} \leq (\sigma^A)^{1-\alpha} \otimes \mathbb{1}^B$.

Using this, we have for $\alpha \in [0, 2] \setminus \{1\}$,

$$E_{RE}^{(\alpha)}(A : B)_\rho \geq \inf_{\sigma^A} \frac{1}{\alpha - 1} \log \text{Tr} [\text{Tr}_B(\rho^{AB})^\alpha] (\sigma^A)^{1-\alpha} \quad (19)$$

$$= \frac{\alpha}{\alpha - 1} \log \text{Tr} [\text{Tr}_B(\rho^{AB})^\alpha]^{1/\alpha} \quad (20)$$

$$= I_\alpha(A)B)_\rho, \quad (21)$$

where the first equality follows from the Sibson's identity (see the supplementary material of Ref. [14]). Arguing similarly by using $\sigma^{AB} \leq \mathbb{1}^A \otimes \sigma^B$, we get $E_{RE}^{(\alpha)}(A : B)_\rho \geq I_\alpha(B)A)_\rho$.

We now prove (18). For a pure state $|\Psi\rangle^{AB}$, $\rho^A = \text{Tr}_B |\Psi\rangle\langle\Psi|^{AB}$, and using (20), we get

$$E_{RE}^{(\alpha)}(A : B)_\rho \geq \frac{\alpha}{\alpha - 1} \log \text{Tr}(\rho^A)^{1/\alpha} = S_{1/\alpha}(A)_\rho. \quad (22)$$

To prove the upper bound, choose $\sigma^{AB} = \sum_i p_i |i\rangle\langle i|^A \otimes |i\rangle\langle i|^B$, where $|\Psi\rangle^{AB} = \sum_i \sqrt{p_i} |i\rangle^A |i\rangle^B$ is the Schmidt decomposition. \square

We now have the following inequality by van Dam and Hayden [22].

Lemma 5 (van Dam and Hayden [22]). *Let $F(\rho, \sigma) \geq F$. For $\alpha \in [0.5, 1)$, the following holds:*

$$S_\alpha(\rho) \geq S_\beta(\sigma) + \frac{2\alpha}{1-\alpha} \log F, \quad (23)$$

where $\beta = \infty$ if $\alpha = 0.5$ and $\beta = \alpha/(2\alpha - 1)$ otherwise.

Using this lemma, we derive an inequality for the RREE.

Lemma 6. *Let ρ^{AB} be a bipartite state and $|\Psi\rangle^{AB}$ be a pure state such that $F(\Psi^{AB}, \rho^{AB}) \geq F$. Then for $\alpha \in (1, 2]$, we have*

$$E_{RE}^{(\alpha)}(A : B)_\rho \geq \frac{2\alpha}{\alpha - 1} \log F + S_{\frac{1}{2-\alpha}}(A)_\Psi. \quad (24)$$

Proof. We first note from (17) in Lemma 4 that

$$E_{RE}^{(\alpha)}(A : B)_\rho \geq I_\alpha(A)_B)_\rho. \quad (25)$$

Define

$$\sigma^A = \frac{\text{Tr}_B(\rho^{AB})^\alpha}{\text{Tr}(\rho^{AB})^\alpha}. \quad (26)$$

Then (25) can be written as

$$E_{RE}^{(\alpha)}(A : B)_\rho \geq S_{\frac{1}{\alpha}}(A)_\sigma - S_\alpha(AB)_\rho. \quad (27)$$

We note that

$$\log F(\sigma^A, \Psi^A) \geq \log F(\sigma^{AB}, \Psi^{AB}) \quad (28)$$

$$= \log F[(\rho^{AB})^\alpha, \Psi^{AB}] - \frac{1}{2} \log \text{Tr}(\rho^{AB})^\alpha \quad (29)$$

$$\geq \alpha \log F(\rho^{AB}, \Psi^{AB}) - \frac{1}{2} \log \text{Tr}(\rho^{AB})^\alpha \quad (30)$$

$$\geq \alpha \log F - \frac{1}{2} \log \text{Tr}(\rho^{AB})^\alpha, \quad (31)$$

where the first inequality follows from the monotonicity under partial trace, the equality follows since $F(\rho, \sigma) = \text{Tr} \sqrt{\sqrt{\sigma} \rho \sqrt{\sigma}}$, the second inequality follows since $\langle \Psi | \rho^\alpha | \Psi \rangle^{AB} \geq (\langle \Psi | \rho | \Psi \rangle^{AB})^\alpha$. Using (31), we have

$$\frac{2}{\alpha - 1} \log F(\sigma^A, \Psi^A) \geq \frac{2\alpha}{\alpha - 1} \log F + S_\alpha(AB)_\rho. \quad (32)$$

We now have from (27)

$$E_{RE}^{(\alpha)}(A : B)_\rho \geq S_{\frac{1}{\alpha}}(A)_\sigma - S_\alpha(AB)_\rho \quad (33)$$

$$\geq \frac{2}{\alpha - 1} \log F(\sigma^A, \Psi^A) + S_{\frac{1}{2-\alpha}}(A)_\Psi - S_\alpha(AB)_\rho \quad (34)$$

$$\geq \frac{2\alpha}{\alpha - 1} \log F + S_{\frac{1}{2-\alpha}}(A)_\Psi, \quad (35)$$

where the second inequality follows from Lemma 5, the third inequality follows from (32) and the claim follows. \square

4 Strong converse for quantum state merging

The following definition of the quantum state merging is almost the same as in Ref. [27].

Definition 1 (State-merging). *Let a pure state $|\Psi\rangle^{ABR}$ be shared between Alice (A) and Bob (B). Let Alice and Bob have quantum registers A_0, A_1 and B_0, B_1 respectively. A (Ψ, F) **state-merging** is a LOCC quantum operation $\mathcal{M} : AA_0 \otimes BB_0 \rightarrow A_1 \otimes B_1 B' B$ such that for $\rho^{A_1 B_1 B' B R} = \mathcal{M}(\Psi^{ABR} \otimes \Phi_K^{A_0 B_0})$,*

$$F\left(\rho^{A_1 B_1 B' B R}, \Phi_L^{A_1 B_1} \otimes \Psi^{B' B R}\right) \geq F. \quad (36)$$

The number $\log K - \log L$ is called the **entanglement cost** of the protocol. In case of many copies $\Psi = \psi^{\otimes n}$, $(\log K - \log L)/n$ is called the **entanglement rate** of the protocol. A real number R is called an **achievable rate** if there exist, for $n \rightarrow \infty$, state-merging protocols of rate approaching R and F approaching 1. The smallest achievable rate is the **merging cost** of ψ .

A fundamental result in quantum information theory is given in the next theorem.

Theorem 7 (Horodecki, Oppenheim and Winter [27]). *For a state $\rho^{AB} = \text{Tr}_R \Psi^{ABR}$ shared by Alice and Bob, the merging cost is the quantum conditional entropy $S(A|B)_\Psi$. If $S(A|B)_\Psi$ is positive, then $R > S(A|B)_\Psi$ ebits are required per input copy and if it is negative, then $R < -S(A|B)_\Psi$ ebits are obtained per input copy by the protocol.*

A converse to the above theorem states that if $\log K - \log L < nS(A|B)_\Psi$, then the fidelity would be bounded away from 1. A strong converse additionally states under the same conditions that the fidelity would go to 0 with n .

A weak converse is provided in Ref. [27]. Strong converse for the entanglement rate for this protocol can be construed through the achievability and strong converse for the Schumacher compression [28]. Berta also provided a strong converse for the entanglement rate in Ref. [29].

4.1 Strong converse with LOCC

We are now ready to provide the converse for the state-merging protocol.

Theorem 8 (Strong converse for the entanglement rate). *For a (Ψ, F) quantum state merging protocol, the following bound holds for $\alpha \in (1, 2]$*

$$\log F \leq n \frac{\alpha - 1}{2\alpha} \left[\frac{\log K - \log L}{n} + S_{2-\alpha}(B)_\Psi - S_{\frac{1}{2-\alpha}}(AB)_\Psi \right]. \quad (37)$$

Proof. We have

$$E_{RE}^{(\alpha)}(AA_0 R : B_0 B)_{|\Psi\rangle^{ABR} \otimes |\Phi_K\rangle^{A_0 B_0}} \leq S_{2-\alpha}(AA_0 R)_{|\Psi\rangle^{ABR} \otimes |\Phi_K\rangle^{A_0 B_0}} \quad (38)$$

$$\leq \log K + S_{2-\alpha}(B)_{|\Psi\rangle}, \quad (39)$$

where the first inequality follows from Lemma 4 and the second one from the additivity of the Rényi entropies for the product states. Let $|\varphi\rangle^{B'BR A_1 B_1} = |\Psi\rangle^{B'BR} \otimes |\Phi\rangle^{A_1 B_1}$. We now have

$$E_{RE}^{(\alpha)}(A_1 R : B_1 B' B)_\rho \geq \frac{2\alpha}{\alpha-1} \log F + S_{\frac{1}{2-\alpha}}(B_1 B' B)_{|\varphi\rangle} \quad (40)$$

$$\geq \frac{2\alpha}{\alpha-1} \log F + \log L + S_{\frac{1}{2-\alpha}}(AB)_{|\Psi\rangle^{ABR}}, \quad (41)$$

where the first inequality follows from Lemma 6. Using the LOCC monotonicity of the RREE, we have

$$E_{RE}^{(\alpha)}(AA_0 R : B_0 B)_{|\Psi\rangle^{ABR} \otimes |\Phi_K\rangle^{A_0 B_0}} \geq E_{RE}^{(\alpha)}(A_1 R : B_1 B' B)_\rho, \quad (42)$$

and hence,

$$\log F \leq \frac{\alpha-1}{2\alpha} \left[\log K - \log L + S_{2-\alpha}(B)_{|\Psi\rangle} - S_{\frac{1}{2-\alpha}}(AB)_{|\Psi\rangle} \right]. \quad (43)$$

For n copies, we have

$$\log F \leq n \frac{\alpha-1}{2\alpha} \left[\frac{\log K - \log L}{n} + S_{2-\alpha}(B)_{|\Psi\rangle} - S_{\frac{1}{2-\alpha}}(AB)_{|\Psi\rangle} \right]. \quad (44)$$

Note that $S_{\frac{1}{2-\alpha}}(AB)_{|\Psi\rangle} - S_{2-\alpha}(B)_{|\Psi\rangle} \leq S(A|B)_{|\Psi\rangle}$ and one can make the LHS approach the RHS by bringing α close to 1 from above. \square

4.2 Strong converse with one-way LOCC

As mentioned in Ref. [27], state-merging can be achieved with just one-way LOCC. Note that since one-way LOCC is a special case of LOCC, therefore, the converse for the entanglement rate remains the same. What we need to provide is the converse for classical communication cost.

Theorem 9 (Strong converse for the classical communication cost). *For a (Ψ, F) quantum state merging protocol, the following bound holds for $\alpha \in (0.5, 1)$ and $\beta = \alpha/(2\alpha - 1)$,*

$$\log F \leq n \frac{1-\alpha}{4\alpha} \left[\frac{\log |X|}{n} - S_\beta(A)_\Psi - S_\beta(R)_\Psi + S_\alpha(AR)_\Psi \right]. \quad (45)$$

Proof. Note that a one-way LOCC could be treated as two cptp maps: the first one by Alice $\mathcal{E} : AA_0 \rightarrow XA_1$, where X is a classical register modeling the classical communication from Alice to Bob, and the second one by Bob $\mathcal{D} : XBB_0 \rightarrow B_1 B' B$.

Let us assume that \mathcal{E} is constructed using an isometry $U_{\mathcal{E}} : AA_0 \rightarrow XA_1 E_1$, where E_1 is the environment and \mathcal{D} is constructed using an isometry $U_{\mathcal{D}} : XBB_0 \rightarrow B_1 B' B E_2$, where E_2 is the environment. Let (with some abuse of notation in the order of the subsystems)

$$|\Xi\rangle^{A_1 X B B_0 E_1} = U_{\mathcal{E}} |\Psi\rangle^{ABR} |\Phi_K\rangle^{A_0 B_0} \quad (46)$$

$$|\Omega\rangle^{A_1 B_1 B' B R E_1 E_2} = U_{\mathcal{D}} \circ U_{\mathcal{E}} |\Psi\rangle^{ABR}. \quad (47)$$

Since

$$F\left(\rho^{A_1 B_1 B' B R}, \Phi_L^{A_1 B_1} \otimes \Psi^{B' B R}\right) \geq F, \quad (48)$$

and $|\Omega\rangle^{A_1 B_1 B' B R E_1 E_2}$ is a purification of $\rho^{A_1 B_1 B' B R}$, using the Uhlmann's theorem [30], there exists a pure state $\kappa^{E_1 E_2}$ such that

$$F(\rho^{A_1 B_1 B' B R}, \Phi_L^{A_1 B_1} \otimes \Psi^{B' B R}) = F(|\Omega\rangle^{A_1 B_1 B' B R E_1 E_2}, \Phi_L^{A_1 B_1} \otimes \Psi^{B' B R} \otimes \kappa^{E_1 E_2}). \quad (49)$$

Using the monotonicity of Fidelity under the partial trace, we have $F(\rho^{A_1 E_1 R}, \Phi_L^{A_1} \otimes \Psi^R \otimes \kappa^{E_1}) \geq F$. Now invoking (92), we have

$$F(\rho^{R A_1 E_1}, \Psi^R \otimes \rho^{A_1 E_1}) \geq F^2. \quad (50)$$

We now have

$$\log |X| \geq S_\beta(X A_1 E_1)_\Xi - S_\beta(A_1 E_1)_\Xi \quad (51)$$

$$\begin{aligned} &\geq S_\beta(X A_1 E_1)_\Xi - S_\beta(A_1 E_1)_\Xi - \\ &\quad [S_\alpha(R X A_1 E_1)_\Xi - S_\alpha(R A_1 E_1)_\Xi] \end{aligned} \quad (52)$$

$$\begin{aligned} &= S_\beta(A A_0)_{\Psi \otimes \Phi_K} - S_\alpha(R A A_0)_{\Psi \otimes \Phi_K} + \\ &\quad S_\alpha(R A_1 E_1)_\rho - S_\beta(A_1 E_1)_\rho \end{aligned} \quad (53)$$

$$= S_\beta(A)_\Psi - S_\alpha(R A)_\Psi + S_\alpha(R A_1 E_1)_\rho - S_\beta(A_1 E_1)_\rho \quad (54)$$

$$\geq S_\beta(A)_\Psi - S_\alpha(R A)_\Psi + S_\beta(R)_\Psi + \frac{4\alpha}{1-\alpha} \log F, \quad (55)$$

where the first two inequalities follows from Lemma 15, the first equality follows since the Rényi entropies are invariant under isometries (the isometry in question is $U_\mathcal{E}$), the second equality follows by canceling out $S_\beta(A_0)_{\Phi_K}$ and using the fact that $S_\beta(A_0)_{\Phi_K} = S_\alpha(A_0)_{\Phi_K}$, the third inequality follows from (92) to have

$$S_\alpha(R A_1 E_1)_\rho \geq S_\beta(R A_1 E_1)_{\Psi \otimes \rho^{A_1 E_1}} + \frac{4\alpha}{1-\alpha} \log F, \quad (56)$$

where we have used the fact that $\rho^R = \Psi^R$.

Since $S_\beta(A)_\Psi + S_\beta(R)_\Psi - S_\alpha(R A)_\Psi \leq I(A : R)_\Psi$ and LHS can be made to approach the RHS by choosing α close to 1 from below. Evaluating the above for n copies gives us the claim of the theorem. \square

5 Strong converse for entanglement concentration

We now provide a short proof for a strong converse for the entanglement concentration protocol which is first defined below.

Definition 2 (Entanglement concentration). Let a pure state $|\Psi\rangle^{AB}$ be shared between Alice (A) and Bob (B). A (Ψ, F) **entanglement concentration** protocol is a LOCC quantum operation $\mathcal{M} : A \otimes B \rightarrow A_1 \otimes B_1$ such that for $\rho^{A_1 B_1} = \mathcal{M}(\Psi^{AB})$,

$$F(\rho^{A_1 B_1}, \Phi_L^{A_1 B_1}) \geq F. \quad (57)$$

In case of many copies $\Psi = \psi^{\otimes n}$, $\log L/n$ is called the rate of the protocol. A real number R is called an **achievable rate** if there exist, for $n \rightarrow \infty$, entanglement concentration protocols of rate approaching R and fidelity approaching 1.

Theorem 10 (Bennett et al [31]). For a pure state Ψ^{AB} shared by Alice and Bob, the highest achievable rate is given by $S(A)_\Psi$. Conversely, any concentration protocol achieving rates higher than $S(A)_\rho$ would have fidelity bounded away from 1.

The strong converse for this protocol follows from the results in Ref. [32]. We now prove a strong converse.

Theorem 11 (Strong converse for entanglement concentration). For any (Ψ, F) entanglement concentration protocol, the following bound holds for $\alpha \in (1, 2]$

$$\log F \leq n \frac{\alpha - 1}{2\alpha} \left[S_{2-\alpha}(A)_\Psi - \frac{\log L}{n} \right]. \quad (58)$$

Proof. We have

$$n S_{2-\alpha}(A)_\Psi \geq E_{RE}^{(\alpha)}(A : B)_\Psi \geq E_{RE}^{(\alpha)}(A_1 : B_1)_\rho \geq \frac{2\alpha}{\alpha - 1} \log F + \log L, \quad (59)$$

where the first inequality follows from Lemma 4, the second inequality follows from Lemma 1 and the third inequality follows from Lemma 6. \square

6 Strong converse for Schumacher compression

We now provide a short proof for a strong converse for the Schumacher compression protocol which is first defined below.

Definition 3 (Schumacher compression). Let Alice have a quantum state ρ^A with purification Ψ^{RA} that she wants to transfer to Bob. A (ρ, F) **Schumacher compression** protocol consists of a cptp compression operation $\mathcal{C} : A \rightarrow B$ (with isometry $U_{\mathcal{C}} : A \rightarrow BE_1$) and a cptp decompression operation $\mathcal{D} : B \rightarrow A_1$ (with isometry $U_{\mathcal{D}} : B \rightarrow A_1 E_2$) with $|\Omega\rangle^{RA_1 E_1 E_2} = U_{\mathcal{D}} \circ U_{\mathcal{C}} |\Psi\rangle^{RA}$ such that

$$F(\Omega^{RA_1}, \Psi^{RA_1}) \geq F. \quad (60)$$

In case of many copies $\rho^{\otimes n}$, $\log |B|/n$ is called the rate of the protocol. A real number R is called an **achievable rate** if there exist, for $n \rightarrow \infty$, Schumacher compression protocols of rate approaching R and fidelity approaching 1.

Theorem 12 (Schumacher [33]). *The smallest achievable rate for Schumacher compression is given by $S(A)_\rho$. Conversely, any Schumacher compression protocol achieving rates smaller than $S(A)_\rho$ would have fidelity bounded away from 1.*

A strong converse for this protocol was first provided by Winter in Ref. [8] and also follows from Ref. [32]. We now prove another strong converse.

Theorem 13 (Strong converse for Schumacher compression). *For any (ρ, F) Schumacher compression protocol, the following bound holds for $\alpha \in (0.5, 1)$ and $\beta = \alpha/(2\alpha - 1)$*

$$\log F \leq n \frac{1 - \alpha}{2\alpha} \left[\frac{\log |B|}{n} - S_\beta(A)_\rho \right]. \quad (61)$$

Proof. Using Uhlmann's theorem [30], there exists a pure state $\tau^{E_1 E_2}$ such that $F(\Omega^{RA_1 E_1 E_2}, \Psi^{RA_1} \otimes \tau^{E_1 E_2}) \geq F$. Using the monotonicity of the Fidelity under partial trace, we get $F(\Omega^{RE_1}, \Psi^R \otimes \tau^{E_1}) \geq F$. We have

$$\log |B| \geq S_\alpha(B)_\Omega = S_\alpha(RE_1)_\Omega \geq S_\alpha(RE_1)_\Omega - S_\beta(E_1)_\tau \quad (62)$$

$$\geq S_\beta(R)_\Psi + \frac{2\alpha}{1 - \alpha} \log F \quad (63)$$

$$= S_\beta(A)_\rho + \frac{2\alpha}{1 - \alpha} \log F, \quad (64)$$

where in the first inequality comes from the fact that a maximally mixed state gives the maximum Rényi entropy, the first equality is because the Rényi entropy does not change under the application of an isometry, the second inequality follows since we are subtracting a non-negative term, the last inequality follows from Lemma 5 and the last equality follows since Ψ^{RA} is pure. Evaluating for n copies $\rho^{\otimes n}$ gives us the claim of the theorem. \square

7 Conclusions and Acknowledgements

To conclude, inspired by the Polyanskiy-Verdú paradigm, we provide strong converses using generalized divergences for protocols involving LOCC maps. While this is illustrated for two protocols using LOCC, nevertheless, the ideas are presented in generality without undue restrictions or made specific to a particular protocol. We provide some inequalities involving the Rényi relative entropy based quantities, in particular, the Rényi relative entropy of entanglement.

We then provide a strong converse for the quantum state merging protocol both for the entanglement rate as well as for the classical communication cost (for one-way LOCC). We also provide a short proof of a strong converse for entanglement concentration of pure states leveraging the earlier developments and for Schumacher compression. It may be possible to improve upon the exponents that we provided.

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A Some more inequalities

The following lemmas are used in the text. The only exception is Lemma 14 which was derived for an approach that got nowhere. We state it here nevertheless.

Lemma 14. *For a cq state $\rho^{RX} = \sum_x p_x \rho_x \otimes |x\rangle \langle x|^X$, $\alpha > 1$, where ρ_x are density matrices and $\{p_x\}$ a probability vector, we have*

$$\log |X| \geq S_\alpha(R)_\rho - S_\alpha(R|X)_\rho, \quad (65)$$

where $S_\alpha(R|X)_\rho \equiv -I_\alpha(X|R)$.

Proof. We have to show that

$$\sum_x p_x (\text{Tr} \rho_x^\alpha)^{1/\alpha} \leq |X|^{(\alpha-1)/\alpha} \left[\text{Tr} \left(\sum_x p_x \rho_x \right)^\alpha \right]^{1/\alpha}. \quad (66)$$

We now have

$$\text{RHS} = \left[|X|^{\alpha-1} \text{Tr} \left(\sum_x p_x \rho_x \right)^\alpha \right]^{1/\alpha} \quad (67)$$

$$\geq \left(|X|^{\alpha-1} \sum_x p_x^\alpha \text{Tr} \rho_x^\alpha \right)^{1/\alpha} \quad (68)$$

$$= \left(\sum_x \frac{1}{|X|} |X|^\alpha p_x^\alpha \text{Tr} \rho_x^\alpha \right)^{1/\alpha} \quad (69)$$

$$\geq \sum_x \frac{1}{|X|} |X| p_x (\text{Tr} \rho_x^\alpha)^{1/\alpha} \quad (70)$$

$$= \text{LHS}, \quad (71)$$

where the first inequality follows from the easy to prove inequality (see Ref. [34] for example) that for positive operators ρ and σ and $\alpha \geq 1$, $\text{Tr}(\rho + \sigma)^\alpha \geq \text{Tr} \rho^\alpha + \text{Tr} \sigma^\alpha$ and the second inequality follows from the concavity of $x \mapsto x^{1/\alpha}$. \square

Lemma 15. *For a cq state ρ^{RX} (classical in X), $\alpha \in [0, 2] \setminus \{1\}$, we have*

$$\log |X| \geq S_\alpha(RX)_\rho - S_\alpha(R)_\rho \geq 0. \quad (72)$$

Proof. Let

$$\rho^{RX} = \sum_x p_x \rho_x \otimes |x\rangle \langle x|^X. \quad (73)$$

We now have

$$S_\alpha(R)_\rho = -\frac{1}{\alpha-1} \log \text{Tr} \left(\sum_x p_x \rho_x \right)^\alpha \quad (74)$$

$$S_\alpha(RX)_\rho = -\frac{1}{\alpha-1} \log \sum_x p_x^\alpha \text{Tr} \rho_x^\alpha. \quad (75)$$

Let $\alpha \in (1, 2)$. To show the left inequality of (72), we have to show that

$$|X|^{\alpha-1} \sum_x p_x^\alpha \text{Tr} \rho_x^\alpha \geq \text{Tr} \left(\sum_x p_x \rho_x \right)^\alpha. \quad (76)$$

We now have

$$\text{RHS} = \text{Tr} \left(\sum_x \frac{1}{|X|} |X| p_x \rho_x \right)^\alpha \quad (77)$$

$$\leq \sum_x \frac{1}{|X|} |X|^\alpha p_x^\alpha \text{Tr} \rho_x^\alpha \quad (78)$$

$$= \text{LHS}, \quad (79)$$

where the inequality follows from the operator convexity of $x \mapsto x^\alpha$.

To show the right inequality of (72), we have to show that

$$\text{Tr} \left(\sum_x p_x \rho_x \right)^\alpha \geq \sum_x p_x^\alpha \text{Tr} \rho_x^\alpha, \quad (80)$$

which follows easily [34].

The case of $\alpha < 1$ is treated similarly. □

Lemma 16. *For density matrices ρ^{AB} , τ^A , σ^B , and $\rho^B = \text{Tr}_A \rho^{AB}$, we have*

$$F(\rho^{AB}, \tau^A \otimes \rho^B) \geq [F(\rho^{AB}, \tau^A \otimes \sigma^B)]^2. \quad (81)$$

Proof. Let us first prove for a pure state $\rho^{AB} = |\Psi\rangle\langle\Psi|^{AB}$. Let $\Psi^B = \text{Tr}_A \Psi^{AB}$ and we have to show that

$$F(\Psi^{AB}, \tau^A \otimes \Psi^B) \geq [F(\Psi^{AB}, \tau^A \otimes \sigma^B)]^2. \quad (82)$$

Let the Schmidt decomposition be $|\Psi\rangle^{AB} = \sum_i \sqrt{\lambda_i} |i\rangle^A |i\rangle^B$. Let $\Psi^A = \text{Tr}_B \Psi^{AB}$. Then $F(\Psi^{AB}, \tau^A \otimes \Psi^B) = \sqrt{\text{Tr} \tau^A (\Psi^A)^2}$ and

$$[F(\Psi^{AB}, \tau^A \otimes \sigma^B)]^2 = \text{Tr} \Psi^{AB} (\tau^A \otimes \sigma^B) \quad (83)$$

$$\leq \text{Tr} \Psi^{AB} (\tau^A \otimes \mathbb{1}^B) \quad (84)$$

$$= \text{Tr} \Psi^A \tau^A \quad (85)$$

$$\leq \sqrt{\text{Tr} \tau^A (\Psi^A)^2} \quad (86)$$

$$= F(\Psi^{AB}, \tau^A \otimes \Psi^B), \quad (87)$$

where the first inequality follows since $\sigma^B \leq \mathbb{1}^B$ and the second inequality is easy to prove.

Now let us prove (81) for mixed ρ^{AB} . Let $\Psi^{R_1 A B R_2}$, $\tau^{R_1 A}$, $\sigma^{B R_2}$ be purifications of ρ^{AB} , τ^A and σ^B respectively such that (invoking the Uhlmann's Theorem [30])

$$F(\rho^{AB}, \tau^A \otimes \sigma^B) = F(\Psi^{R_1 A B R_2}, \tau^{R_1 A} \otimes \sigma^{B R_2}). \quad (88)$$

Let $\Psi^{B R_2} = \text{Tr}_{R_1 A} \Psi^{R_1 A B R_2}$ and we make no assumption that it is pure. We now have

$$F(\rho^{AB}, \tau^A \otimes \rho^B) \geq F(\Psi^{R_1 A B R_2}, \tau^{R_1 A} \otimes \Psi^{B R_2}) \quad (89)$$

$$\geq [F(\Psi^{R_1 A B R_2}, \tau^{R_1 A} \otimes \sigma^{B R_2})]^2 \quad (90)$$

$$= [F(\rho^{AB}, \tau^A \otimes \sigma^B)]^2, \quad (91)$$

where the first inequality follows from the monotonicity of the fidelity under partial trace, the second inequality follows from (82), and the equality at the end follows from our choice of purifications.

In particular, choosing $\tau^A = \text{Tr}_B \rho^{AB} \equiv \rho^A$ yields

$$F(\rho^{AB}, \rho^A \otimes \rho^B) \geq [F(\rho^{AB}, \rho^A \otimes \sigma^B)]^2. \quad (92)$$

This inequality would be put to use later. QED. \square

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